need not be the total profit; it may be the total production, or whatever other quantity we wish to take as a desirable characteristic of the social system we discuss. The author regrets that at the present time he can refer only to his lecture courses for a further treatment of this point of view. Nevertheless it seems the most fruitful way that a really theoretical economics may be developed.

ON KELLOGG'S DIOPHANTINE PROBLEM.[1]

By D. R. CURTISS, Northwestern University.

1. Kellogg's Problem. Two Applications. In a recent number of the MONTHLY,[2] Professor Kellogg has presented a very interesting discussion of the Diophantine equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}=1, \tag{1}$$

in which he gives reasons for believing that the maximum value of any of the unknowns that can occur in a solution in positive integers is u_n, where

$$u_1 = 1, \qquad u_{k+1} = u_k(u_k + 1). \tag{2}$$

Thus the successive u's are 1, 2, 6, 42, 1806, $\cdots$. I propose here to give a proof of the correctness of this statement, a proof in which we use sequences of inequalities, each containing one less x than the preceding. The method may be of some interest in itself, and of some value in similar problems. That the proof is hardly so simple as the statement of the problem will cause no surprise, at least to one familiar with Diophantine analysis.

Before we take up this proof it may add some interest to note two problems, one geometrical, the other arithmetical, whose solution depends on finding particular sets of integers satisfying (1). The first is that of laying non-overlapping sets of floor-tiles, each tile being a regular polygon whose sides are of unit length, so as to cover the plane, or a portion of the plane, just once or multiply; the polygons of a set will not, in general, all have the same number of sides. For example, suppose we are to fit n such tiles against each other so that all shall have a common vertex, and the piece of surface formed by them shall wind k times about this vertex, the last tile being in such a position as to fit without overlapping against the first tile if it were in the first layer instead of the last. In other words, the tiles are to generate without overlapping a piece of a Riemann surface in which the sheets form one cycle about a branch point. If the n tiles

[1] Read before the American Mathematical Society, December, 1921.

[2] *1921*, 300–303. See references there to Carmichael's *Diophantine Analysis* and to his review of Dickson's *History of the Theory of Numbers*, vol. 2, in this MONTHLY. This subject is very close to that of Sylvester's paper, "On a point in the theory of vulgar fractions," *American Journal of Mathematics*, vol. 3, 1880, pp. 332–335 and 388–389, which, Sylvester says, was suggested by the account in Cantor's *Geschichte der Mathematik* of the ancient Egyptian treatment of fractions by resolution into a sum of fractions each having unity as its numerator.

are regular polygons of $m_1, m_2, \cdots, m_n$ sides, respectively, so that their interior angles are

$$\pi - \frac{2\pi}{m_1}, \quad \pi - \frac{2\pi}{m_2}, \quad \cdots, \quad \pi - \frac{2\pi}{m_n},$$

the sum of these angles must be $2k\pi$. From this we deduce the relation

$$\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_n} = \frac{n}{2} - k.$$

If n is even we divide through by $(n/2) - k$ and are thus led to seek solutions of (1) in which the x's have a common factor $(n/2) - k$. If n is odd we may add $\frac{1}{2}$ to both sides of the above equation and reduce to form (1) by dividing through by $((n + 1)/2) - k$. Another reduction to (1) will suggest itself if we require all the m's to be even. We are thus led to consider solutions of (1) under certain restrictions.

The other problem is that of finding perfect numbers, a positive integer being defined as perfect if it is equal to the sum of all its different divisors less than itself, including unity. Thus 6 is a perfect number since $6 = 1 + 2 + 3$. For even perfect numbers we have a formula ascribed to Euclid, but it is not known whether an odd perfect number exists.[1] Let a_n be a perfect number, so that

$$a_n = 1 + a_1 + a_2 + \cdots + a_{n-1},$$

where $1, a_1, a_2, \cdots, a_{n-1}$ are all the divisors of a_n, not including a_n itself. The quotient of a_n by any a is an a, so, if we divide the above equation by a_n and rearrange, we have

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} + \frac{1}{a_n} = 1.$$

Since the largest a is a_n itself, the correctness of Kellogg's statement for (1) has this consequence:

A perfect number with n divisors less than itself (unity included) cannot be greater than the number u_n defined by (2).

This result may be of some use in discussing the existence of an odd perfect number. We may lower this upper bound for an odd perfect number by trying to find the maximum x in a solution of (1) where all the x's are required to be odd and unequal.

Both these applications suggest the discussion of solutions of (1) under restrictions, and especially the determination of the maximum x under such added conditions. Problems of this type present difficulties, and I shall not attempt their discussion; perhaps some reader will be interested to try his fortune in this field.

2. A Theorem which Includes Kellogg's. Let us return now to the problem we have undertaken to solve. Where no ambiguity is thereby introduced we

[1] Compare this MONTHLY, *1921*, 140–141.

shall find it convenient to use the symbol $f_r(x)$ defined by

$$\frac{1}{f_r(x)} = 1 - \frac{1}{x_1} - \frac{1}{x_2} - \cdots - \frac{1}{x_r}. \tag{3}$$

With this notation, (1) can be put in the form

$$x_n = f_{n-1}(x).$$

The result which we are to prove, and which includes Kellogg's theorem, we will state as follows:

THEOREM I. *The maximum finite value of $f_{n-1}(x)$ for all positive integral values of $x_1, x_2, \cdots, x_{n-1}$ is u_n as defined by (2). There is but one set of x's which gives this maximum value, namely that in which*

$$x_k = u_k + 1, \qquad k = 1, 2, \cdots, n-1.$$

Note that we are not here restricting ourselves (as in Kellogg's formulation) to values of the x's that make $f_{n-1}(x)$ integral; we find, however, that the maximum of this expression under the given conditions is an integer.

The reader will find in Kellogg's paper a proof that the set $x_k = u_k + 1$ actually gives to $f_{n-1}(x)$ the value u_n.

3. Necessary Conditions for a Maximum. Reduced Sets. As a first step in our proof, a glance at (3) shows that if all but one of the x's are fixed, then the best choice for the remaining x, *i.e.*, the value that makes $f_{n-1}(x)$ as large as possible, but finite, is the least value of that x for which $1/f_{n-1}(x)$ is positive. Thus we should take x_{n-1} as the least integer such that

$$1 - \frac{1}{x_1} - \frac{1}{x_2} - \cdots - \frac{1}{x_{n-1}} > 0,$$

i.e., such that

$$x_{n-1} > f_{n-2}(x).$$

If we use the symbol $E(a)$ to denote the greatest integer which does not exceed a, this gives us as our best choice for x_{n-1},

$$x_{n-1} = E(f_{n-2}(x)) + 1. \tag{4}$$

In particular, *a necessary condition for a maximum of $f_{n-1}(x)$ is that the x's be a set in which the member or one of the members having the largest value, taken as x_{n-1}, verifies (4).*

In a set which has the above property we cannot decrease any one member, leaving the others unaltered, without making $1/f_{n-1}(x)$ negative or zero; let us call such a set a *compact set.* When a set is compact, x_{n-1} being the largest member, we have

$$f_{n-1}(x) \geqq x_{n-1}(x_{n-1} - 1),$$

for this is equivalent to

$$\frac{1}{f_{n-1}(x)} \leqq \frac{1}{x_{n-1} - 1} - \frac{1}{x_{n-1}}, \qquad \text{or} \qquad 1 - \frac{1}{x_1} - \frac{1}{x_2} - \cdots - \frac{1}{x_{n-1} - 1} \leqq 0,$$

which follows from the definition of a compact set.

We will use the term *reduced set* for a compact set which has but one largest member. Let us consider two ways of transforming any given set of integers $x_1, x_2, \cdots, x_r$ for which $1/f_r(x)$ is positive into a reduced set $x_1', x_2', \cdots, x_r'$. The first step, naturally, is to take the largest x (or one of them if there are equal largest x's) and decrease it to the smallest value which will make $1/f_r(x)$ positive, while the other x's are left unaltered; if a largest member of the resulting set can be decreased we proceed as in the first instance, repeating the process until we obtain a compact set $X_1, X_2, \cdots, X_{r-1}, X_r$, which we will suppose arranged in order of magnitude. In case $X_{r-1} = X_r$, two methods of procedure are suggested by formulas indicated by Kellogg. In the first we replace the set X by the set Y in which

$$Y_1 = 2, \quad Y_2 = 2X_1, \quad Y_3 = 2X_2, \quad \cdots, \quad Y_{r-1} = 2X_{r-2}, \quad Y_r = X_r. \tag{5}$$

It is easily verified that

$$f_r(Y) = 2f_r(X).$$

We now make the Y set compact by the process already described. Then, if the resulting set has equal largest members, apply (5) so as to obtain a set Z. If these operations are continued long enough we must finally obtain a reduced set. In fact, a set x whose first k members are $2, 4, 8, \cdots, 2^k$ cannot contain any other member as small as 2^k without making $1/(f_r(x))$ zero or negative. But the set Y contains one member 2, the set Z one member 2 and one 4, and so on, and if we have not at some previous stage obtained a reduced set we shall have transformed our set into the set $2, 4, 8, \cdots, 2^r$, and by making this compact we have the reduced set $2, 4, 8, \cdots, 2^{r-1}, 2^{r-1} + 1$. Note that *every step has increased* $f_r(x)$.

The other method of reduction proceeds from the identities[1]

$$\frac{2}{x} \equiv \frac{1}{\frac{x}{2}+1} + \frac{1}{\frac{x}{2}\left(\frac{x}{2}+1\right)}, \qquad \frac{2}{x} \equiv \frac{1}{\frac{x+1}{2}} + \frac{1}{x\left(\frac{x+1}{2}\right)} \tag{6}$$

If a compact set X has two largest members equal, so that $X_{r-1} = X_r$, we replace the set X by the set

$$X_1, X_2, \cdots, X_{r-2}, \frac{X_r}{2}+1, \frac{X_r}{2}\left(\frac{X_r}{2}+1\right), \tag{7}$$

when X_r is even, and by

$$X_1, X_2, \cdots, X_{r-2}, \frac{X_r+1}{2}, X_r\left(\frac{X_r+1}{2}\right), \tag{8}$$

when X_r is odd; from (6) we see that the new set gives to $f_r(x)$ the same value as does the set X. We then make the new set compact as before; and this compact set will also be a reduced set. For by (7) or (8) we have replaced two equal numbers X_r by two integers, one less than X_r and the other greater. If we could reduce the latter to be less than or equal to one of the other X's, we should have a set in which no number is greater than the corresponding X and at least one

[1] The first of these is given in Kellogg's paper.

number is less, which is impossible, since the set X was compact by hypothesis.

As an illustration, consider the set 3, 4, 5, 6. The first method transforms this into 3, 4, 5, 5 (compact), and then into 2, 6, 8, 5, which, rearranged, is 2, 5, 6, 8 (reduced). By the second method we obtain successively 3, 4, 5, 5, then 3, 3, 4, 15, and finally 3, 3, 4, 13 (reduced).

The set (7) or (8) gives to $f_r(x)$ the *same* value as the set X, so that we cannot say for the second method, as we did for the first, that at every step we increase $f_r(x)$. Whether a set (7) or (8), derived from a compact set X for which $X_{r-1} = X_r$, may ever be compact I have not been able to determine.[1] Until that point is settled we cannot say that in every case the reduced set obtained by the second method gives to $f_r(x)$ a greater value than the original set. This is, however, true of the first method, so that we may strengthen the condition for a maximum given at the beginning of this section as follows:

THEOREM II. *A necessary condition that a set x maximize $f_{n-1}(x)$ is that it be a reduced set; i.e.; we can so choose our notation that we have*

$$x_1 \leqq x_2 \leqq \cdots \leqq x_{n-2} < x_{n-1}, \tag{9}$$
$$x_{n-1} = E(f_{n-2}(x)) + 1. \tag{4}$$

We assume here and in what follows that $n > 2$, the case $n = 2$ being easily disposed of.

4. The Function $\phi_{n-2}(x)$. Necessary Conditions for Maximizing $\phi_{n-2}(x)$. Having restricted ourselves to reduced sets x, we can now assign an upper bound for $f_{n-1}(x)$. In fact, this last function is expressible as a fraction whose numerator is the product $x_1x_2 \cdots x_{n-1}$, and whose denominator is a positive integer. We thus have

$$f_{n-1}(x) \leqq x_1x_2 \cdots x_{n-1} \tag{10}$$

where x_{n-1} is given by (4), and all the x's satisfy (9).

Since

$$E(f_{n-2}(x)) + 1 \leqq f_{n-2}(x) + 1,$$

we can replace (9) and (10) by the inequalities

$$x_1 \leqq x_2 \leqq \cdots \leqq x_{n-2} \leqq f_{n-2}(x), \tag{11}$$
$$f_{n-1}(x) \leqq x_1x_2 \cdots x_{n-2}[f_{n-2}(x) + 1]. \tag{12}$$

We will denote the second member of (12) by $\phi_{n-2}(x)$.

The next step is to investigate the maximizing of $\phi_{n-2}(x)$ for sets $x_1, x_2, \cdots, x_{n-2}$ subject to (11). We shall prove the following result:

THEOREM III. *A necessary condition that a set $x_1, x_2, \cdots, x_{n-2}$ subject to (11) maximize $\phi_{n-2}(x)$ is that it be a reduced set.*

To prove this, let us start with any set verifying (11) which makes $1/f_{n-2}(x)$

[1] While the sets (7) and (8) are not usually compact, they are so in some cases. For example, 7, 8, 10, 10, 10, 11, 11, 12, 12, 12, is compact, and the set (7) derived from it, 7, 8, 10, 10, 10, 11, 11, 12, 7, 42, is also compact. In a similar manner, 9, 9, 10, 11, 11, 11, 11, 12, 13, 13, 13, is compact as is also the corresponding set (8), namely, 9, 9, 10, 11, 11, 11, 11, 12, 13, 7, 91. For this second example $f_{11}(x)$ has the remarkably large value of 25,740—EDITOR.

positive and transform this set into a reduced set by the second method. We will show that at each step we have a set verifying (11) and giving to $\phi_{n-2}(x)$ a value greater than it had before. The former of these statements is easily disposed of, for our second method employs two kinds of processes. The first process decreases one element at a time, which increases $f_{n-2}(x)$ and leaves (11) true when the symbols are suitably arranged. The second substitutes (7) or (8) for a compact set X with two largest members equal. It is, then, only the effect of this latter substitution that we need examine; we give the details for X_{n-2} even. The last member of (7) is the largest, and $f_{n-2}(x)$ is the same for the set (7) as for the set X, so that we have to prove

$$\frac{X_{n-2}}{2}\left(\frac{X_{n-2}}{2}+1\right) \leqq f_{n-2}(x).$$

But since X is compact $f_{n-2}(X) \geqq X_{n-2}(X_{n-2}-1)$, and since X_{n-2} must be greater than 2, we have $X_{n-2}(X_{n-2}-1) > (X_{n-2}/2)\ ((X_{n-2}/2)+1)$. Therefore the new set satisfies (11). We have a similar result when X_{n-2} is odd, depending on the fact that here $X_{n-2} > 3$.

Having shown that each transformed set for our second method satisfies (11), we must now prove that each step increases $\phi_{n-2}(x)$. These steps consisted either in decreasing one member at a time, or in passing from a compact set X with two equal largest members to a set (7) or (8). To prove that a step of the former sort increases the value of $\phi_{n-2}(x)$, suppose we have a set x satisfying (11), which we transform into a set x' by changing only the member x_{n-2}; this latter we replace by $x'_{n-2} = x_{n-2} - 1$. We suppose $f_{n-2}(x')$ positive. Since

$$\frac{1}{f_{n-2}(x)} - \frac{1}{f_{n-2}(x')} = \frac{1}{x_{n-2}(x_{n-2}-1)},$$

we have

$$f_{n-2}(x') = \frac{x_{n-2}(x_{n-2}-1)f_{n-2}(x)}{x_{n-2}(x_{n-2}-1)-f_{n-2}(x)},$$

and $\phi_{n-2}(x') - \phi_{n-2}(x)$ is equal to $x_1x_2 \cdots x_{n-3}$ multiplied by

$$\begin{aligned}(x_{n-2}-1)[f_{n-2}(x')+1] &- x_{n-2}[f_{n-2}(x)+1] \\ &= \frac{x_{n-2}(x_{n-2}-1)^2f_{n-2}(x)}{x_{n-2}(x_{n-2}-1)-f_{n-2}(x)} - x_{n-2}f_{n-2}(x) - 1 \\ &= \frac{x_{n-2}[f_{n-2}(x)]^2 - (x_{n-2}^2 - x_{n-2} - 1)f_{n-2}(x) - x_{n-2}(x_{n-2}-1)}{x_{n-2}(x_{n-2}-1)-f_{n-2}(x)}.\end{aligned}$$

The denominator of this fraction is positive, being the denominator of the expression for $f_{n-2}(x')$. The numerator is a quadratic in $f_{n-2}(x)$ with both roots less than x_{n-2}. For if we substitute x_{n-2} for $f_{n-2}(x)$ it reduces to $2x_{n-2}$, which is positive. Therefore the numerator of the fraction is positive and

$$\phi_{n-2}(x') > \phi_{n-2}(x).$$

It remains only to show that the substitution of a set (7) or (8) for a compact

set X in which $X_{n-3} = X_{n-2}$ gives to $\phi_{n-2}(x)$ a value greater than $\phi_{n-2}(X)$. We carry this through for the case where X_{n-2} is even; the reader will easily see how to treat the other case. The substitution to be carried out leaves $f_{n-2}(x)$ unaltered, and in fact the only change in $\phi_{n-2}(x)$ is to substitute for the two equal factors X_{n-2} in $\phi_{n-2}(X)$ the pair $X_{n-2}/2 + 1$, $(X_{n-2}/2)\,(X_{n-2}/2 + 1)$. But the product of this pair is greater than $X_{n-2}{}^2$, for $X_{n-2} > 2$, so that

$$\left(\frac{X_{n-2}}{2} - 1\right)^2 > 0, \qquad \left(\frac{X_{n-2}}{2} + 1\right)^2 > 2X_{n-2}, \qquad \frac{X_{n-2}}{2}\left(\frac{X_{n-2}}{2} + 1\right)^2 > X_{n-2}{}^2.$$

Thus $\phi_{n-2}(x)$ has been increased by this substitution.

Since each step in our second method, which must terminate with a reduced set, increases $\phi_{n-2}(x)$, while all the sets used satisfy (11), we have proved Theorem III.

5. Proof of Theorem I. If we now take $x_1, x_2, \cdots, x_{n-2}$ as a reduced set, so that

$$x_{n-2} = E(f_{n-3}(x)) + 1 \leqq f_{n-3}(x) + 1,$$

we obtain by comparison with (10), (11), and (12) the inequalities

$$x_1 \leqq x_2 \leqq \cdots \leqq x_{n-3} \leqq f_{n-3}(x), \tag{13}$$

$$f_{n-1}(x) \leqq \phi_{n-2}(x) \leqq x_1x_2 \cdots x_{n-2}(x_1x_2 \cdots x_{n-2} + 1) \leqq \phi_{n-3}(x)[\phi_{n-3}(x) + 1]. \tag{14}$$

To maximize the right member of (14) is to maximize $\phi_{n-3}(x)$, and we are thus led to repetitions of section 4. Our final inequalities are

$$x_1 \leqq f_1(x), \qquad f_{n-1}(x) \leqq U_{n-2}, \tag{15}$$

where U_{n-2} is defined by the recurrence relations

$$U_1 = \phi_1(x),\, U_{k+1} = U_k(U_k + 1), \qquad k = 1, 2, \cdots, n-3.$$

But (15) gives us $x_1 \leqq 1/(1 - (1/x_1))$, $x_1 = 2$, since $x_1 < 2$ is impossible. With this value for x_1 we have

$$U_1 = \phi_1(x) = x_1\left(\frac{1}{1 - \dfrac{1}{x_1}} + 1\right) = 6 = u_3, \qquad \text{and} \qquad U_{n-2} = u_n,$$

to use the notation of (2). Thus we have

$$f_{n-1}(x) \leqq u_n.$$

But the value u_n is actually attained by giving to the x's the values $u_k + 1$, so that u_n is the maximum of $f_{n-1}(x)$.

One way of stating the necessary conditions at each reduction of our problem was to require that each of the sets,

$$x_1; \quad x_1, x_2; \quad \cdots; \quad x_1, x_2, \cdots, x_{n-1},$$

be reduced. The only x's that satisfy these conditions are those of the set $x_k = u_k + 1$. With this remark our proof of Theorem I is complete.

AUTHOR'S NOTE—Since the above was put into type, the author has received reprints of an article by Tanzô Takenouchi, entitled "On an indeterminate equation," which has just appeared in the *Proceedings of the Physico-Mathematical Society of Japan* (third series, volume 3, pp. 78–92). In this paper the number 1 on the right side of equation (1) is replaced by a fraction b/a in which $a \geqq b$, and in case a is of the form $(m+1)b-1$, it is shown that the maximum x in all solution sets ($n > 1$) is A_n, as given by the recurrence formula

$$A_1 = m, \qquad A_2 = a(A_1+1), \qquad \cdots, \qquad A_{k+1} = A_k(A_k+1).$$

When $b = m = 1$, this is Kellogg's theorem (see p. 91). The results of the present paper are thus anticipated in a general way, though Theorem I is not proved by Takenouchi. The methods employed are sufficiently different to make the present paper of interest, and its greater simplicity and brevity may recommend it. The methods we have here used apply with very little change to Takenouchi's problem. We cannot, in general, use the first method of reduction for a compact set with equal largest members, so that the necessity of some of our conditions does not follow, but we easily obtain Takenouchi's result. We can, in fact, go further than he has, and obtain the following analogue of Theorem I, where

$$\frac{1}{f_r(x)} = \frac{b}{a} - \frac{1}{x_1} - \frac{1}{x_2} - \cdots - \frac{1}{x_r},$$

and $b \leqq a$:

The maximum finite value of $f_{n-1}(x)$, $n > 1$, for all positive integral values of x_1, $x_2, \cdots x_{n-1}$ is not greater than B_n, where

$$B_1 = E\left(\frac{a}{b}\right),\ B_2 = a(B_1+1), \qquad B_3 = B_2\left(\frac{1}{\frac{b}{a} - \frac{a}{B_2}} + 1\right),$$

$$B_{k+1} = B_k(B_k+1),\ k > 2.$$

If $a = (m+1)b-1$, we have $B_n = A_n$, so that this upper bound is actually reached.

In Takenouchi's example

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{5}{11},$$

which does not come under the case where he has given a general solution but for which he observed that the maximum x is at least 220, the upper bound noted above is $305\frac{1}{4}$.